

Large-scale density from velocity expansion and shear

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ABSTRACT

I derive up to second order in Eulerian perturbation theory a new relation between the weakly nonlinear density and velocity fields. In the case of unsmoothed fields, density at a given point turns out to be a purely local function of the expansion (divergence) and shear of the velocity field. The relation depends on Ω , strongly by the factor $f(\Omega) \simeq \Omega^{0.6}$ and weakly by the factors $K(\Omega) \propto \Omega^{-2/63}$ and $C(\Omega) \propto \Omega^{-1/21}$. The Gramann solution is found to be equivalent to the derived relation with the weak Ω -dependence neglected. To make the relation applicable to the real world, I extend it for the case of smoothed fields. The resulting formula, when averaged over shear given divergence, reproduces up to second order the density–velocity divergence relation of Chodorowski & Łokas; however, it has smaller spread. It makes the formula a new attractive local estimator of large-scale density from velocity.

Key words: cosmology: theory – galaxies: clustering – galaxies: formation – large-scale structure of the Universe

1 INTRODUCTION

The value of Ω remains still one of the most intriguing unknowns in cosmology today. The parameter Ω , defined as the ratio of the mean to the critical density is so crucial for cosmology because its value determines the global geometry and ultimate fate of Universe.

One way to measure Ω is to compare large-scale density fields of galaxies with the corresponding fields of galaxy velocities. Under widely accepted hypothesis of gravitational instability, observed large-scale peculiar flows of galaxies (deviations from Hubble flow) result from gravitational growth of initially small cosmic mass fluctuations. The quantitative relation between the mass density contrast field, $\delta = \rho/\langle\rho\rangle - 1$, where $\langle\rho\rangle$ is the mean density, and the peculiar velocity field, \mathbf{v} , can be deduced from the dynamical equations describing the pressureless self-gravitating cosmic fluid. For small density fluctuations linear theory can be applied. Linear theory predicts that the density contrast is a linear and local function of the velocity *divergence*,

$$\delta^{(1)}(\mathbf{x}) = -f(\Omega)^{-1} \nabla \cdot \mathbf{v}^{(1)}(\mathbf{x}). \quad (1)$$

In the above, the function $f(\Omega) \simeq \Omega^{0.6}$ (see e.g. Peebles 1980) and the superscript ‘(1)’ denotes the linear theory limit. (Distances are measured here in km s^{-1} , so the Hubble constant $H = 1$ in this system of units.) The above formula can be used to reconstruct from a large-scale velocity field the linear mass density field, up to an Ω -dependent multiplicative factor $f(\Omega)$. The comparison of the reconstructed mass field with the observed large-scale galaxy density field

may therefore serve as a method for estimating Ω and as a test for the gravitational instability hypothesis.

Indeed, a strong correlation between the galaxy density and velocity divergence fields has been found in observations (Dekel et al. 1993; Hudson et al. 1995; Sigad et al. 1997). However, equation (1) assumes linear theory while the fields in question are weakly nonlinear. Smoothing of the fields, necessary among other things to reduce large individual distance-estimation errors and the shot noise content, must be performed over a limited scale in order to optimize the information present in the finite-volume data. The POTENT algorithm for the mass density reconstruction from an observed radial velocity field currently employs a Gaussian smoothing length of 1000–1200 km s^{-1} (Dekel 1994; Dekel et al. 1997). At these scales, typical (rms) galaxy density fluctuations are of the order of several tens per cent, in contradiction with an underlying assumption of equation (1) that $\delta \ll 1$. On the other hand, they are not in excess of unity, wherefore the name “weakly nonlinear”. The need for a weakly nonlinear generalization of linear formula (1) has been quickly recognized. The present POTENT algorithm uses the formula of Nusser et al. (1991), which is the Zel’dovich (1970) approximation expressed in Eulerian coordinates. However, N-body simulations (Mancinelli et al. 1994; Ganon et al. 1997) have shown that though Zel’dovich approximation does much better than linear theory equation (1), it still does not predict correctly the weakly nonlinear density–velocity relation (hereafter DVR).

Weakly nonlinear regime is the regime of applicability of perturbation theory. To begin with, linear theory solutions for the density and velocity divergence fields that give rise to

linear equation (1) are nothing but perturbative series truncated at the lowest, i.e. first order terms. A natural way of extending linear DVR into weakly nonlinear regime is thus to take into account higher order perturbative contributions for density and velocity divergence. This has been done by Chodorowski & Lokas (1997a, hereafter CL97), who computed weakly nonlinear density–velocity divergence relation up to third order in perturbation theory. (Second order contributions were included already by Bernardeau 1992.) The resulting extension of the linear formula offers also a method for separating the effects of Ω and possible bias between galaxy and mass distributions (CL97; Bernardeau, Chodorowski & Lokas 1997, hereafter BCL).

One might worry that the perturbative approximation to nonlinear DVR breaks down soon after the linear relation does so. However, N-body simulations (BCL; Ganon et al. 1997; Chodorowski et al. 1997) show the opposite: the perturbative formula is a very good robust fit to N-body results in the whole cosmologically interesting range of smoothing radii.

Higher order perturbative solutions for density and velocity divergence are nonlocal. As a result, the relation between weakly nonlinear density and velocity divergence *at a given point* is no longer deterministic. Still, since the spread comes exclusively from higher order contributions the two fields remain strongly correlated and the mean trend can serve as a useful local approximation to the true nonlocal DVR. This is exactly what has been calculated by CL97, who found that the formula for the conditional mean density given velocity divergence is given by the third-order polynomial in velocity divergence. The reverse case (mean velocity given density) has been calculated by Chodorowski & Lokas (1997b). Work is in progress on the theoretical prediction for the spread (Chodorowski, Lokas & Pollo 1997), as well as its measurement in N-body simulations (BCL; Chodorowski et al. 1997).

Summarizing the above in the statistical language, the perturbative polynomial in velocity divergence is an unbiased but non-zero variance local estimator of density. It is then natural to ask a question: among all unbiased local estimators of density from velocity, is it the minimum-variance one? It is quite unlikely. The fact that in weakly nonlinear regime density and velocity divergence at a given point are not related in a deterministic way does not exclude a possible existence of a purely local formula for density as a function of some derivatives of the velocity field, $v_{i,j}$. The irrotationality of the flow implies only that the tensor of velocity derivatives is symmetric, $v_{i,j} = v_{j,i}$. Hence, it has 6 independent components, while the velocity divergence involves only 3 diagonal ones out of them.

Indeed, already mentioned Zel’dovich approximation which does involve off-diagonal components is, like linear formula (1), deterministic. The formula based on Zel’dovich approximation is a biased estimator, but it is not based on rigorous Eulerian perturbation theory. The application of perturbation theory and the inclusion of the off-diagonal components in the velocity derivatives tensor may result in a local estimator which is both unbiased and has smaller variance than the relation of CL97. Why not simply zero-variance? To make the estimator applicable to the real world, smoothing of the fields has to be included. Smoothing is a nonlocal operation, hence it necessarily introduces

some spread into any nonlinear DVR; I will discuss this in more detail later on. Still, since the density–velocity divergence relation is nonlocal already for unsmoothed fields, for smoothed fields it can be expected to have greater spread than the corresponding relation which is local when unsmoothed.

This paper is devoted to constructing such a lower-variance estimator of density from velocity. In section 2 I derive up to second order in Eulerian perturbation theory a purely local relation between unsmoothed density and velocity fields. In section 3 I relate it to the Gramann (1993) solution. Subsequently, in section 4 I derive from it the unsmoothed density–velocity divergence relation. In section 5 I generalize my relation for the case of smoothed fields. Section 6 is devoted to the comparison of the relation to N-body simulations. Finally, I summarize the results in section 7.

2 DERIVATION

Let us expand the density contrast in a perturbative series,

$$\delta = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \dots \quad (2)$$

In the above, $\delta^{(p)}$ denotes the p -th order perturbative contribution, which is of the order of $(\delta^{(1)})^p$ (Fry 1984; Goroff et al. 1986). Introduce a variable proportional to the velocity divergence

$$\vartheta(\mathbf{x}, t) \equiv -f(\Omega)^{-1} \nabla_{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t) \quad (3)$$

and expand it as well,

$$\vartheta = \vartheta^{(1)} + \vartheta^{(2)} + \vartheta^{(3)} + \dots \quad (4)$$

The linear theory solution, equation (1), gives

$$\delta^{(1)}(\mathbf{x}) = \vartheta^{(1)}(\mathbf{x}). \quad (5)$$

From equations (2), (4) and (5) we have up to second order

$$\delta(\mathbf{x}) = \vartheta(\mathbf{x}) + \delta^{(2)}(\mathbf{x}) - \vartheta^{(2)}(\mathbf{x}). \quad (6)$$

The second order perturbative contributions for δ and ϑ for arbitrary Ω are (Bouchet et al. 1992; Bernardeau et al. 1995)

$$\delta^{(2)} = \frac{1+K}{2}(\delta^{(1)})^2 + \nabla_{\mathbf{x}}\delta^{(1)} \cdot \nabla_{\mathbf{x}}\Phi^{(1)} + \frac{1-K}{2}\Phi_{,ij}^{(1)}\Phi_{,ij}^{(1)} \quad (7)$$

and

$$\vartheta^{(2)} = C(\delta^{(1)})^2 + \nabla_{\mathbf{x}}\delta^{(1)} \cdot \nabla_{\mathbf{x}}\Phi^{(1)} + (1-C)\Phi_{,ij}^{(1)}\Phi_{,ij}^{(1)}. \quad (8)$$

Here, $\Phi^{(1)}(\mathbf{x}, t)$ is the linear gravitational potential satisfying the Poisson equation

$$\Delta_{\mathbf{x}}\Phi^{(1)} = \delta^{(1)} \quad (9)$$

and I use the Einstein summation convention. The weakly Ω -dependent functions K and C are

$$K(\Omega) = \frac{3}{7}\Omega^{-2/63} \quad (10)$$

and

$$C(\Omega) = \frac{3}{7}\Omega^{-1/21}. \quad (11)$$

The approximation (10) is accurate to within 0.4 per cent in the range $0.05 < \Omega < 3$ (Bouchet et al. 1992) and (11) to

within 2 per cent in the range $0.1 < \Omega < 10$ (Bernardeau et al. 1995).

Subtracting equation (8) from (7) we have

$$\delta^{(2)} - \vartheta^{(2)} = \frac{1+K-2C}{2}(\vartheta^{(1)})^2 - \frac{1+K-2C}{2}\Phi_{,ij}^{(1)}\Phi_{,ij}^{(1)}. \quad (12)$$

Note that the terms $\nabla_{\mathbf{x}}\delta^{(1)} \cdot \nabla_{\mathbf{x}}\Phi^{(1)}$ have cancelled out. The well-known solution to equation (9) for the gravitational potential is

$$\Phi^{(1)}(\mathbf{x}) = - \int \frac{d^3\mathbf{x}'}{4\pi} \frac{\delta^{(1)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \quad (13)$$

Hence, the difference $\delta^{(2)}(\mathbf{x}) - \vartheta^{(2)}(\mathbf{x})$, in addition to the local term $\sim (\vartheta^{(1)})^2(\mathbf{x})$, contains a nonlocal term due to the gravitational potential, $\sim \sum \alpha_{\mathbf{x}'\mathbf{x}''}\delta^{(1)}(\mathbf{x}')\delta^{(1)}(\mathbf{x}'') \sim \sum \alpha_{\mathbf{x}'\mathbf{x}''}\vartheta^{(1)}(\mathbf{x}')\vartheta^{(1)}(\mathbf{x}'')$. The right-hand side of equation (12) is thus not a function, but a functional of ϑ . On the one hand, this means that up to second order in perturbation theory the velocity divergence field still contains full information about the density field. On another hand, however, this information is distributed non-locally, that is the value of the density contrast at a given point \mathbf{x} is only determined by the values of the velocity divergence in all space. Shortly, the relation between the weakly nonlinear density and velocity divergence fields is deterministic, but nonlocal. Such a relation is of no use for cosmology, since we have at our disposal only surveys of limited volume, with incomplete sky coverage, uneven sampling, etc. The key point is to find a local relation between the density and velocity fields.

One of the ways to do this is to approximate the nonlocal-in- ϑ term in equation (12) by its conditional mean, given the velocity divergence at a point \mathbf{x} (Bernardeau 1992; CL97). The resulting density–velocity divergence relation is then local by construction but clearly not deterministic; in other words it has some spread. The idea of the present paper is different: to recast the nonlocal-in- ϑ term to the form involving other local derivatives of the velocity field.

Since the velocity field in weakly nonlinear regime remains irrotational it can be expressed as a gradient of some velocity potential $\Phi_{\mathbf{v}}$,

$$\mathbf{v} = -\nabla_{\mathbf{x}}\Phi_{\mathbf{v}}. \quad (14)$$

Taking the divergence of the above equation and using equation (1) shows that in linear regime the velocity potential is proportional to the gravitational potential,

$$\Phi_{\mathbf{v}}^{(1)} = f(\Omega)\Phi^{(1)}, \quad (15)$$

and equations (14)–(15) yield

$$\Phi_{,ij}^{(1)} = -f^{-1}v_{i,j}^{(1)}. \quad (16)$$

Let us decompose the symmetric tensor of velocity derivatives into its trace (expansion), θ , and the traceless part (shear), σ_{ij} ,

$$v_{i,j} = \frac{1}{3}\theta\delta_{ij} + \sigma_{ij}, \quad (17)$$

where in general

$$\sigma_{ij} \equiv \frac{1}{2}(v_{i,j} + v_{j,i}) - \frac{1}{3}\theta\delta_{ij} \quad (18)$$

and

$$\theta \equiv \nabla_{\mathbf{x}} \cdot \mathbf{v} = v_{k,k}. \quad (19)$$

The symbol δ_{ij} denotes the Kronecker’s delta. Note by comparing with equation (3) that

$$\vartheta = -f^{-1}\theta. \quad (20)$$

We have

$$v_{i,j}v_{i,j} = \frac{1}{3}\theta^2 + \sigma^2, \quad (21)$$

where σ^2 is the shear scalar

$$\sigma^2 = \sigma_{ij}\sigma_{ij}. \quad (22)$$

From equations (16) and (21) taken at linear order we obtain

$$\Phi_{,ij}^{(1)}\Phi_{,ij}^{(1)} = \frac{1}{3}f^{-2}(\theta^{(1)})^2 + f^{-2}(\sigma^{(1)})^2. \quad (23)$$

Introducing the above equation to equation (12) and using (20) yields

$$\begin{aligned} \delta^{(2)} - \vartheta^{(2)} &= \frac{1+K-2C}{3}f^{-2}(\theta^{(1)})^2 \\ &- \frac{1+K-2C}{2}f^{-2}(\sigma^{(1)})^2. \end{aligned} \quad (24)$$

Denote the rms fluctuation of the linear density field by ε , $\varepsilon^2 = \langle (\delta^{(1)})^2 \rangle$. Perturbation theory predicts that $\theta^{(p)} \sim \delta^{(p)} \sim \varepsilon^p$ so we have $\theta^2 = (\theta^{(1)} + \theta^{(2)} + \dots)^2 = (\theta^{(1)})^2 + \mathcal{O}(\varepsilon^3)$ and similarly for σ^2 . Therefore, constructing weakly nonlinear density–velocity relation up to terms quadratic in ε we can substitute $(\theta^{(1)})^2$ by θ^2 and $(\sigma^{(1)})^2$ by σ^2 . By doing so and introducing equation (24) into (6) we finally obtain

$$\begin{aligned} \delta(\mathbf{x}) &= -f^{-1}\theta(\mathbf{x}) + \frac{1+K-2C}{3}f^{-2}\theta^2(\mathbf{x}) \\ &- \frac{1+K-2C}{2}f^{-2}\sigma^2(\mathbf{x}) + \mathcal{O}(\varepsilon^3). \end{aligned} \quad (25)$$

This is the DVR computed up to second order in perturbation theory. The density is a local function of the expansion and the shear of the velocity field. A similar relation has been independently derived by Catelan et al. (1995), though without the decomposition of the term $v_{i,j}v_{i,j}$ into the expansion and the shear (cf. eq. [41] of Catelan et al. 1995). The relation depends on Ω , strongly by the factor f and weakly by the factors K and C given by equations (10)–(11). Neglecting the weak Ω -dependence we obtain

$$\delta(\mathbf{x}) = -f^{-1}\theta(\mathbf{x}) + \frac{4}{21}f^{-2}\theta^2(\mathbf{x}) - \frac{2}{7}f^{-2}\sigma^2(\mathbf{x}). \quad (26)$$

Formula (25) for density involves only first derivatives of the velocity field. We owe this to the fortunate cancellation of the terms $\nabla_{\mathbf{x}}\delta^{(1)} \cdot \nabla_{\mathbf{x}}\Phi^{(1)}$ in equation (12). Had not these terms cancelled, we would have had on the right hand side of equation (25) an extra contribution $\sim \nabla_{\mathbf{x}}\Phi^{(1)} \cdot \nabla_{\mathbf{x}}\theta^{(1)} \sim v_{j,i}v_{i,j} + \mathcal{O}(\varepsilon^3)$, involving the second derivatives. The practical applicability of such a formula to extracting density from a very noisy velocity field would be doubtful.

3 RELATION TO THE GRAMANN SOLUTION

Gramann (1993) derived DVR up to second order in Lagrangian perturbation theory. The form of this relation expressed in Eulerian coordinates is

$$\delta(\mathbf{x}) = -f^{-1}\theta(\mathbf{x}) + \frac{4}{7}f^{-2}m_{\mathbf{v}}(\mathbf{x}), \quad (27)$$

where

$$m_v = \sum_{i < j}^3 (v_{i,i} v_{j,j} - v_{i,j} v_{j,i}), \quad (28)$$

or, explicitly

$$\begin{aligned} m_v &= v_{1,1} v_{2,2} + v_{1,1} v_{3,3} + v_{2,2} v_{3,3} \\ &- v_{1,2} v_{2,1} - v_{1,3} v_{3,1} - v_{2,3} v_{3,2}. \end{aligned} \quad (29)$$

In the following I will relate equation (27) to my solution for weakly nonlinear DVR calculated in the previous section. Let us decompose the tensor of velocity derivatives more generally into the expansion, shear, and vorticity (the asymmetric part), respectively,

$$v_{i,j} = \frac{1}{3} \theta \delta_{ij} + \sigma_{ij} - \frac{1}{2} \epsilon_{ijk} \omega_k. \quad (30)$$

Here, ϵ_{ijk} is the completely antisymmetric tensor; $\epsilon_{123} = 1$. After applying the above decomposition, a few lines of algebra yield

$$\begin{aligned} v_{1,1} v_{2,2} + v_{1,1} v_{3,3} + v_{2,2} v_{3,3} &= \sigma_{11} \sigma_{22} + \sigma_{11} \sigma_{33} + \sigma_{22} \sigma_{33} \\ &+ \frac{1}{3} \theta^2. \end{aligned} \quad (31)$$

Similarly,

$$v_{1,2} v_{2,1} + v_{1,3} v_{3,1} + v_{2,3} v_{3,2} = \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 - \frac{1}{4} \omega^2, \quad (32)$$

where ω^2 is the vorticity scalar,

$$\omega^2 \equiv \boldsymbol{\omega} \cdot \boldsymbol{\omega} = \omega_k \omega_k. \quad (33)$$

By definition,

$$\sigma^2 = \sigma_{ij} \sigma_{ij} = 2 (\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) + \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2. \quad (34)$$

Using the above equation and the identity

$$(\sigma_{11} + \sigma_{22} + \sigma_{33})^2 = 0, \quad (35)$$

we can recast equation (32) to the form

$$\begin{aligned} v_{1,2} v_{2,1} + v_{1,3} v_{3,1} + v_{2,3} v_{3,2} &= \sigma_{11} \sigma_{22} + \sigma_{11} \sigma_{33} + \sigma_{22} \sigma_{33} \\ &+ \frac{1}{2} \sigma^2 - \frac{1}{4} \omega^2. \end{aligned} \quad (36)$$

Using equations (31), (36) and (29) we obtain

$$m_v = \frac{1}{3} \theta^2 - \frac{1}{2} \sigma^2 + \frac{1}{4} \omega^2. \quad (37)$$

The final step is to use this result in equation (27), the result is

$$\begin{aligned} \delta(\mathbf{x}) &= -f^{-1} \theta(\mathbf{x}) + \frac{4}{21} f^{-2} \theta^2(\mathbf{x}) - \frac{2}{7} f^{-2} \sigma^2(\mathbf{x}) \\ &+ \frac{1}{7} f^{-2} \omega^2(\mathbf{x}). \end{aligned} \quad (38)$$

Thus, the weakly nonlinear density in a given point is in general determined by the local values of the three scalars that can be constructed from the derivatives of the velocity field: the expansion, shear and vorticity. Since before shell crossing a cosmic velocity field is irrotational we can in the above equation drop out the vorticity term. Then the equation coincides exactly with equation (26) of the previous section. Equation (26) is the DVR up to second order in perturbation theory with the weak Ω -dependence neglected. Summarizing, the Gramann (1993) solution (27)

is the second-order DVR with only the strong Ω -dependence included.

Equation (38) bears some resemblance to the Raychaudhuri (1955) differential equation for the evolution of the velocity expansion. There, the source terms are similarly proportional to density, expansion, shear and vorticity (cf. eq. [22.14] of Peebles 1980).

4 DENSITY—VELOCITY DIVERGENCE RELATION

As already mentioned in section 2, density and velocity divergence at a given point are not related in a deterministic way. Let us rewrite equation (25) in the form

$$\begin{aligned} \delta(\mathbf{x}) &= \vartheta(\mathbf{x}) + \frac{1+K-2C}{3} \vartheta^2(\mathbf{x}) - \frac{1+K-2C}{2} \Sigma^2(\mathbf{x}) \\ &+ \mathcal{O}(\varepsilon^3), \end{aligned} \quad (39)$$

where ϑ is related to θ by equation (20) and I have defined

$$\Sigma_{ij} \equiv -f^{-1} \sigma_{ij}. \quad (40)$$

The spread in the δ - ϑ relation clearly comes from the shear. The mean trend, defined as mean δ given ϑ , is

$$\langle \delta \rangle|_{\vartheta} = \vartheta + \frac{1+K-2C}{3} \vartheta^2 - \frac{1+K-2C}{2} \langle \Sigma^2 \rangle|_{\vartheta} + \mathcal{O}(\varepsilon^3). \quad (41)$$

By definition of the conditional moment,

$$\langle \Sigma^2 \rangle|_{\vartheta} = \frac{\int \Sigma^2 p(\vartheta, \Sigma) d\Sigma}{p(\vartheta)}, \quad (42)$$

where $p(\vartheta, \Sigma)$ is the joint probability distribution function (PDF) for expansion and the shear scalar. It is sufficient to know the form of this PDF for linear ϑ and Σ since already at the lowest order $\Sigma^2 \sim \varepsilon^2$ and nonlinear corrections are $\sim \mathcal{O}(\varepsilon^3)$.

How to derive the joint distribution for $\vartheta^{(1)}$ and $\Sigma^{(1)}$? In the derivation of its general properties I will follow Juszkiewicz et al. (1995; Appendix A). I assume that the initial conditions are Gaussian. Under this assumption, both $\vartheta^{(1)}$ and five independent components of the shear tensor $\Sigma_{ij}^{(1)}$ are Gaussian distributed. Consequently, $p(\vartheta^{(1)}, \Sigma_{ij}^{(1)})$ is a multivariate Gaussian, entirely determined by its covariance matrix. It is more convenient to compute the coefficients of this matrix in the Fourier space. The Fourier transform of $\vartheta^{(1)}$ is obviously equal to the Fourier transform of the linear density field, $\vartheta_{\mathbf{k}}^{(1)} = \delta_{\mathbf{k}}^{(1)}$. Thereafter I will drop out the superscripts ‘(1)’. The power spectrum $P(k)$ is defined by the relation

$$\langle \delta_{\mathbf{k}} \delta_{\mathbf{k}'} \rangle = (2\pi)^3 \delta_D(\mathbf{k} + \mathbf{k}') P(k), \quad (43)$$

where δ_D denotes the Dirac’s delta. The Fourier transform of a shear component is

$$(\Sigma_{ij})_{\mathbf{k}} = \left(\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij} \right) \delta_{\mathbf{k}}, \quad (44)$$

where $\hat{k}_i \equiv k_i/k$. We have

$$\begin{aligned} \langle \vartheta \Sigma_{ij} \rangle &= \int \frac{d^3 k d^3 k'}{(2\pi)^6} \left(\hat{k}_i \hat{k}_j' - \frac{1}{3} \delta_{ij} \right) \langle \delta_{\mathbf{k}} \delta_{\mathbf{k}'} \rangle e^{-i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{x}} \\ &= \int \frac{d^3 k}{(2\pi)^3} \left(\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij} \right) P(k) \\ &= 0. \end{aligned} \quad (45)$$

The second step uses equation (43) and the last one is obvious by symmetry. It means that the linear shear components are uncorrelated with the linear velocity divergence. In general, the uncorrelation of random variables is only a necessary condition for their statistical independence; for Gaussian variables however it is also the sufficient one. Therefore, ϑ and Σ_{ij} are statistically independent; consequently the variables ϑ and $\Sigma = (\Sigma_{ij}\Sigma_{ij})^{1/2}$ are also statistically independent. It implies

$$p(\vartheta, \Sigma) = p(\vartheta)p(\Sigma). \quad (46)$$

Using this result in equation (42) yields

$$\langle \Sigma^2 \rangle|_{\vartheta} = \int \Sigma^2 p(\Sigma) d\Sigma, \quad (47)$$

that is the mean value of Σ^2 does not depend on ϑ and is equal to the ordinary mean. We have

$$\langle \Sigma_{ij}\Sigma_{ij} \rangle = \langle \Phi_{,ij}\Phi_{,ij} \rangle - \frac{1}{3}\langle \delta^2 \rangle. \quad (48)$$

The integration by parts of the term $\langle \Phi_{,ij}\Phi_{,ij} \rangle$ yields $\langle \Phi_{,ii}\Phi_{,jj} \rangle = \langle \delta^2 \rangle = \varepsilon^2$, hence

$$\langle \Sigma^2 \rangle = \frac{2}{3}\varepsilon^2. \quad (49)$$

(cf. Silk 1974). Introducing this in equation (41) we finally obtain

$$\langle \delta \rangle|_{\vartheta} = \vartheta + a_2(\vartheta^2 - \varepsilon^2) + \mathcal{O}(\varepsilon^3), \quad (50)$$

where

$$a_2 = \frac{1 + K - 2C}{3}. \quad (51)$$

Equations (50)-(51) agree with the results of CL97 obtained by a completely different method, namely the Edgeworth expansion of the joint PDF for the variables δ and ϑ . Specifically, formula (51) for the coefficient a_2 coincides with equation (81) of CL97 (for the case of no smoothing).

I will now compute the spread around the mean trend, or the conditional variance. Hereafter in this section I will neglect the weak dependence of equations (39) and (41) on Ω . We then have

$$\langle (\delta - \langle \delta \rangle|_{\vartheta})^2 \rangle|_{\vartheta} = \frac{4}{49} \left\langle (\Sigma^2 - \langle \Sigma^2 \rangle|_{\vartheta})^2 \right\rangle|_{\vartheta} + \mathcal{O}(\varepsilon^6). \quad (52)$$

It is known that Σ^2 is χ^2 -distributed with 5 degrees of freedom; the variance of the Gaussian variable underlying the distribution is $2/15\varepsilon^2$ (Górski 1988; cf. also Groth, Juszkiewicz & Ostriker 1989). Therefore

$$\left\langle (\Sigma^2 - \langle \Sigma^2 \rangle|_{\vartheta})^2 \right\rangle|_{\vartheta} = 2 \cdot 5 \cdot \frac{4}{225}\varepsilon^4 = \frac{8}{45}\varepsilon^4, \quad (53)$$

hence finally

$$\langle (\delta - \langle \delta \rangle|_{\vartheta})^2 \rangle|_{\vartheta} = \frac{32}{2205}\varepsilon^4 + \mathcal{O}(\varepsilon^6). \quad (54)$$

The above formula exactly coincides with the result of Chodorowski, Łokas & Pollo (1997; for the case of no smoothing), obtained by means of the Edgeworth expansion.

And if initial conditions are non-Gaussian? In this case property (45) still holds as its derivation does not require any assumption about initial conditions. Therefore, ϑ and Σ_{ij} remain uncorrelated. It however does not necessarily

mean that they are statistically independent. Uncorrelated non-Gaussian variables may be, but equally well may not be, statistically independent; see Kendall & Stuart (1973) for the examples of both cases. Catelan et al. (1997) show that for a certain class of non-Gaussian models ϑ and Σ are indeed independent. For this class of non-Gaussian models the density-velocity divergence relation is therefore the same as for Gaussian initial conditions, equation (50). Only the spread around the mean trend is different since Σ^2 is no longer χ^2 -distributed. The form of the δ - ϑ relation for other non-Gaussian models remains to be investigated.

5 EFFECTS OF SMOOTHING

Ganon et al. (1997) test various approximations for weakly nonlinear DVR by the means of N-body simulations. Among the approximations considered is the Gramann (1993) solution. Since the Gramann solution is equivalent to equation (25) with the weak Ω -dependence neglected (section 3), its properties essentially reflect the properties of the present solution. Ganon et al. (1997) plot the difference between the approximate and true density as a function of the true density, $\mathcal{D} = \mathcal{D}(\delta)$. The Gramann solution gives significant residuals which have a parabolic form. Is equation (25) thus incorrect? There is certainly no error in its derivation, but it cannot be straightforwardly applied to the *smoothed* density and velocity fields that were estimated from N-body by Ganon et al. (1997).

Inferring the fields from observations one has to introduce smoothing. The first reason for doing so is to reduce the effects of large individual distance-estimation errors and the shot noise. The second one is that only the field of radial peculiar velocities is directly measured; to reconstruct from it the full three-dimensional velocity field we need to assume its potentiality. This assumption is valid only before trajectory crossing (where the Kelvin's circulation theorem holds), that is for the smoothed fields. However, even if we were provided by Nature with an accurate 3D velocity field (as accessible in N-body simulations) we would still prefer to smooth it, in order to reduce the nonlinearities. We would do so because linear or weakly nonlinear density and velocity fields are strongly correlated and the form of this correlation offers us a possibility to test the gravitational instability hypothesis and to measure the value of Ω .

Smoothing of the fields is realised by averaging them with a certain window of a certain scale R , W_R . For example, the smoothed density contrast field is

$$\bar{\delta}(\mathbf{x}) = \int d^3x' \delta(\mathbf{x}') W_R(\mathbf{x} - \mathbf{x}'). \quad (55)$$

The velocity fields are commonly smoothed with a Gaussian filter. In this case, the second-order density-velocity divergence relation still has the form (50), but the coefficient a_2 is a function of the power spectrum index n ,

$$a_2 = \frac{1 + K - 2C}{3} {}_2F_1\left(\frac{n+3}{2}, \frac{n+3}{2}, \frac{5}{2}, \frac{1}{4}\right). \quad (56)$$

Here, ${}_2F_1$ is the hypergeometric function (CL97). The effective power-law index n is the slope of the log P -log k relation at the smoothing scale R (Bernardeau 1994). For $n = -3$ the coefficient a_2 is $(1 + K - 2C)/3$, equal to that for the case of

no smoothing, equation (51). The factor $(1 + K - 2C)/3$ can be well approximated by its value for $\Omega = 1$: $4/21 \simeq 0.19$. For higher spectral indices a_2 grows monotonically up to the value $\simeq 0.30$ for $n = 1$. At the scales of interest, i.e. of several megaparsecs, the effective index of the *observed* power spectrum is clearly different from the value $n = -3$; e.g. for *IRAS* galaxies it is $n = -1.4$ (Fisher et al. 1993). Also, the power spectra which are commonly used in N-body simulations have the values of the effective index different from $n = -3$; e.g. for the standard CDM, for $R > 5h^{-1}$ Mpc, $n \geq -1$ (e.g. CL97). Therefore, equation (50), when applied to the smoothed fields, underestimates density because the value of the coefficient a_2 is underestimated. Specifically,

$$\mathcal{D} = \Delta a_2(\delta^2 - \varepsilon^2) + \mathcal{O}(\varepsilon^3), \quad (57)$$

where $\Delta a_2 \equiv a_2(-3) - a_2(n)$. The residual \mathcal{D} is thus a parabola in δ with a negative coefficient. Finally, relation (25), equivalent to the Gramann solution, must yield the same residual, since δ - ϑ relation (50) is its version averaged over the shear. This is indeed observed in the simulations: the Gramann solution yields essentially the same parabolic residual as the δ - ϑ approximation of Bernardeau (1992), derived also for the case of no smoothing.

How to extend equation (25), or its another form (39), for the case of smoothed fields? Applying a smoothing filter to both sides of equation (39) we have

$$\begin{aligned} \bar{\delta}(\mathbf{x}) &= \bar{\vartheta}(\mathbf{x}) + \frac{1+K-2C}{3} \bar{\vartheta}^2(\mathbf{x}) - \frac{1+K-2C}{2} \bar{\Sigma}^2(\mathbf{x}) \\ &+ \mathcal{O}(\varepsilon^3), \end{aligned} \quad (58)$$

where e.g. the smoothed $\bar{\vartheta}^2$ is given by

$$\bar{\vartheta}^2(\mathbf{x}) = \int d^3x' \vartheta^2(\mathbf{x}') W_R(\mathbf{x} - \mathbf{x}'). \quad (59)$$

However, from observations we can estimate only smoothed fields, and only after smoothing we can perform transformations on them (like squaring). Our purpose is therefore to express the right-hand side of equation (58) as a function of $\bar{\vartheta}$ and

$$\bar{\Sigma} \equiv (\bar{\Sigma}_{ij} \bar{\Sigma}_{ij})^{1/2}. \quad (60)$$

Since smoothing and nonlinear transformations do not commute, in general $\bar{\vartheta}^2$ is not equal to $\bar{\vartheta}^2$; similarly for the shear. It is most clearly seen by writing

$$\bar{\vartheta}^2 = \overline{(\vartheta - \bar{\vartheta})^2} + \bar{\vartheta}^2 \geq \bar{\vartheta}^2, \quad (61)$$

wherefrom $\bar{\vartheta}^2$ is equal to $\bar{\vartheta}^2$ only when the spectral index $n = -3$, i.e. when the fluctuations have so large wavelengths that $\bar{\vartheta} = \vartheta$. Moreover, the same values of $\bar{\vartheta}$ can lead to different values of $\bar{\vartheta}^2$. (As a simplest academic example the reader can consider one-dimensional fields $\vartheta_1 = 1$ and $\vartheta_2 = 2x$, smoothed with a top-hat filter over the segment $[0, 1]$.) It means that the relation between $\bar{\delta}(\mathbf{x})$ and the variables $\bar{\vartheta}(\mathbf{x})$ and $\bar{\Sigma}(\mathbf{x})$ is non-deterministic. Again, the mean trend is given by the conditional mean,

$$\langle \bar{\delta} \rangle_{\bar{\vartheta}, \bar{\Sigma}} = \bar{\vartheta} + \frac{1+K-2C}{3} \left\langle \bar{\vartheta}^2 - \frac{3}{2} \bar{\Sigma}^2 \right\rangle_{\bar{\vartheta}, \bar{\Sigma}}. \quad (62)$$

The standard approach would be to derive the joint PDF for the four variables $\bar{\vartheta}^2$, $\bar{\Sigma}^2$, $\bar{\vartheta}$, and $\bar{\Sigma}$, $p(\bar{\vartheta}^2, \bar{\Sigma}^2, \bar{\vartheta}, \bar{\Sigma})$, and to integrate over it the second term on the right-hand

side of the above equation. Fortunately, this horrible calculation is unnecessary because the result can be simply guessed. Firstly, it must be a quadratic form in $\bar{\vartheta}$ and $\bar{\Sigma}$, since it comes from second-order perturbative contributions. Second, when averaged over $\bar{\Sigma}$, it must reduce to the second term of the smoothed version of equation (50),

$$\langle \bar{\delta} \rangle_{\bar{\vartheta}} = \bar{\vartheta} + a_2(\bar{\vartheta}^2 - \bar{\varepsilon}^2), \quad (63)$$

where a_2 is given by equation (56) and $\bar{\varepsilon}^2 \equiv \langle \bar{\varepsilon}^2 \rangle$.

I postulate that

$$\frac{1+K-2C}{3} \left\langle \bar{\vartheta}^2 - \frac{3}{2} \bar{\Sigma}^2 \right\rangle_{\bar{\vartheta}, \bar{\Sigma}} = a_2 \left(\bar{\vartheta}^2 - \frac{3}{2} \bar{\Sigma}^2 \right). \quad (64)$$

The first condition is obviously satisfied. Similarly to equation (45), we have

$$\begin{aligned} \langle \bar{\vartheta} \bar{\Sigma}_{ij} \rangle &= \int \frac{d^3k}{(2\pi)^3} \left(\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij} \right) P(k) W_R^2(k) \\ &= 0. \end{aligned} \quad (65)$$

Here, $W_R(k)$ is the Fourier transform of the window function. Thus, also the smoothed fields $\bar{\vartheta}$ and $\bar{\Sigma}$ are statistically independent. It implies that $\langle \bar{\Sigma}^2 \rangle_{\bar{\vartheta}} = \langle \bar{\Sigma}^2 \rangle = (2/3) \bar{\varepsilon}^2$ (eq. [49]), hence the second condition is satisfied as well. Finally, an additional term $\propto \bar{\vartheta} \bar{\Sigma}$ on the right-hand-side of equation (64) would violate it, because $\langle \bar{\vartheta} \bar{\Sigma} \rangle_{\bar{\vartheta}} = \bar{\vartheta} \langle \bar{\Sigma} \rangle \neq 0$: the average of the shear scalar does not, unlike the average of its component $\bar{\Sigma}_{ij}$, vanish since $\bar{\Sigma}$ is positive-definite.

Thus, the form on the right-hand-side of equation (64) is the unique quadratic form in $\bar{\vartheta}$ and $\bar{\Sigma}$ which, when averaged over $\bar{\Sigma}$, reduces to the second term of equation (63). Therefore, postulated equation (64) is indeed correct. Combining it with (62) we obtain

$$\langle \bar{\delta} \rangle_{\bar{\vartheta}, \bar{\Sigma}} = \bar{\vartheta} + a_2 \left(\bar{\vartheta}^2 - \frac{3}{2} \bar{\Sigma}^2 \right), \quad (66)$$

or, using equations (20) and (40),

$$\langle \bar{\delta} \rangle_{\bar{\vartheta}, \bar{\sigma}} = -f^{-1} \bar{\theta} + a_2 f^{-2} \left(\bar{\theta}^2 - \frac{3}{2} \bar{\sigma}^2 \right). \quad (67)$$

Here, $\bar{\theta}$ and $\bar{\sigma}$ is the expansion and shear of the *smoothed* velocity field $\bar{\mathbf{v}}$,

$$\bar{\theta} \equiv \nabla_{\mathbf{x}} \cdot \bar{\mathbf{v}} \quad (68)$$

and

$$\bar{\sigma} \equiv (\bar{\sigma}_{ij} \bar{\sigma}_{ij})^{1/2} \quad (69)$$

with

$$\bar{\sigma}_{ij} \equiv \frac{1}{2} (\bar{v}_{i,j} + \bar{v}_{j,i}) - \frac{1}{3} \bar{\theta} \delta_{ij}. \quad (70)$$

Equation (66), with the coefficient a_2 given by equation (56), constitutes an extension of equation (39) for the case of smoothed fields. As already discussed, smoothing of the fields induces spread in the relation between the smoothed density and the smoothed expansion and shear in a given point. In the present paper I will not attempt to compute the spread. However, it is certainly smaller than the spread in the smoothed density-velocity divergence relation (63), since this relation is obtained from (66) by averaging over the shear. This averaging is an extra source of the spread in the δ - ϑ relation: for unsmoothed fields the spread

is given by equation (54), while the relation (39) between density and the two velocity scalars is entirely deterministic.

It should be stressed that equation (66) assumes Gaussian initial conditions, since the coefficient a_2 , equation (56), has been computed by CL97 under this assumption. For non-Gaussian initial conditions, a detailed form of the relation between the smoothed density and the smoothed expansion and shear remains to be studied.

6 COMPARISON TO N-BODY SIMULATIONS

Ganon et al. (1997) have performed N-body simulations for a CDM family of models. For the fields smoothed with a Gaussian window of radius $R = 12h^{-1}$ Mpc, Ganon et al. have estimated the value of the coefficient a_2 to be

$$a_2^{(NB)} \simeq 0.28 \quad (71)$$

(cf. also Willick et al. 1997). As explained in the previous section, the Gramann approximation yields

$$a_2^{(G)} = \frac{4}{21} \simeq 0.19. \quad (72)$$

What is the value of a_2 predicted by the approximation derived here? The hypergeometric function in expression (56) can be expanded in powers of $n + 3$, simply by rearranging the appropriate terms of the hypergeometric series (cf. Lokas et al. 1995). In the expansion, the term linear in $n + 3$ vanishes. The resulting formula for the coefficient a_2 , truncated at the cubic term, is

$$a_2 = \frac{4}{21} [1 + c_2(n + 3)^2 + c_3(n + 3)^3], \quad (73)$$

where

$$c_2 = 0.02139 \quad \text{and} \quad c_3 = 0.00370. \quad (74)$$

The polynomial (73) approximates expression (56) in the range $-3 \leq n \leq 1$ with accuracy of 0.2% or better. If $\Omega \neq 1$, the factor $4/21$ should be replaced by the factor $(1 + K - 2C)/3$, weakly varying with Ω .

For a smoothing radius of $12h^{-1}$ Mpc, the effective index of the standard CDM spectrum is (for details see CL97)

$$n = -0.404. \quad (75)$$

Using formula (73), the predicted value of the coefficient a_2 is then

$$a_2 \simeq 0.23. \quad (76)$$

This value lies half-way between $a_2^{(G)}$ of the Gramann approximation, equation (72), and $a_2^{(NB)}$ estimated from N-body by Ganon et al. (1997), equation (71). It means, on the one hand, that the density–velocity relation derived in this paper, equation (67) with a_2 given by equation (56) or (73), is not in full agreement with the results of N-body by Ganon et al. On the plot of the difference between the approximate and the true density as a function of the true density, inferred from N-body, it will still leave some parabolic residuals. On the other hand, however, these residuals will be smaller, roughly two times, than for the Gramann approximation. As stated earlier, the Gramann approximation is valid only for unsmoothed fields. Not surprisingly then, the

proper second-order formula for smoothed fields is a better estimator of smoothed density from smoothed velocity.

The slight discrepancy between the value (76) and that estimated from N-body can be attributed to perturbative contributions of higher orders. Here, or in the paper of CL97, the value of a_2 has been derived at the lowest relevant order in perturbation theory, the second. Higher-order corrections yield a contribution to a_2 which is of the order of $\bar{\epsilon}^2 = \langle \bar{\delta}^2 \rangle$. In other words,

$$a_2(\bar{\epsilon}) = a_2(\bar{\epsilon} \rightarrow 0) + p\bar{\epsilon}^2 + \mathcal{O}(\bar{\epsilon}^4), \quad (77)$$

where the dimensionless coefficient p remains to be calculated. Since the (linear) variance of the field in question (Gaussian smoothed with radius of $12h^{-1}$ Mpc) is $\bar{\epsilon}^2 = 0.076$, the value $p \sim 0.7$ is sufficient to account for the discrepancy between a_2 and $a_2^{(NB)}$.

On the other hand, it is important to confirm the results of Ganon et al. (1997) by independent simulations. Preliminary results of such show that the coefficient a_2 has indeed slightly higher value than predicted by second order perturbation theory, approaching it for small $\bar{\epsilon}$ (BCL; Chodorowski et al. 1997).

7 SUMMARY

In the present paper I have studied the relation between the weakly nonlinear cosmic density and velocity fields. I have derived up to second order in perturbation theory an expression for density as a purely local function of the expansion (divergence) and shear of the velocity field (eq. [25] or [39]). The relation depends on Ω both strongly and weakly. I have shown that the Gramann (1993) solution (eq. [27]) is equivalent to equation (25) with the weak Ω -dependence neglected. Also, a similar relation has been independently found by Catelan et al. (1995).

The locality of the relation derived here is in contrast with the non-locality of the density–velocity divergence relation calculated by CL97. I have shown that averaging of equation (39) over shear given divergence yields up to second order the formula of CL97 (eq. [50]); thus, the source of the spread in the latter relation is the distribution of shear. Since the form of this distribution is known, it is straightforward to compute the higher-order conditional moments of the density–velocity divergence relation. In particular, I have computed the spread of this relation (the conditional variance; eq. [54]) and have found it to coincide with the result of rather cumbersome calculations by Chodorowski, Lokas & Pollo (1997).

Smoothing is a necessary ingredient of large-scale density–velocity comparisons. I have then generalized equation ([25]) for the case of smoothed fields (eq. [67] with a_2 given by eq. [56]). Smoothing not only modifies the shape of the relation between density and the two velocity scalars but also induces the spread in it. I have not computed the spread explicitly. Still, I have shown that it is smaller than the corresponding spread in the density–velocity divergence relation, since that spread has an extra source: averaging over shear given divergence.

Equation (25) for unsmoothed fields does not depend on the type of initial conditions. Its smoothed counterpart (67), however, assumes Gaussian initial conditions; a

detailed form of it for non-Gaussian models remains to be investigated.

Checks against N-body simulations show that relation (67) is still a slightly biased estimator of large-scale density from velocity (section 6). However, it was not the main concern of this paper. The ultimate goal of attempts to establish semi-linear relations between the density and velocity fields is to construct a local estimator of density from velocity which is not only unbiased but has minimum variance as well. The present paper is the first attempt to address the second point. As stated earlier, the inclusion of the shear term in the relation between the large-scale density and velocity reduces its spread. In this paper I have demonstrated how to include the shear in the second-order perturbative relation. In future, I plan to do so for higher-order relations as well.

The relation density versus velocity expansion and shear, derived here, is a new, lower-variance estimator of density from the velocity field that can be applied in the cosmic density-velocity comparisons.

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